

PSEUDOCIRCLES IN DYNAMICAL SYSTEMS

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ABSTRACT. We construct an example of a C^∞ map on a 3-manifold which has an invariant set with an uncountable number of components, each of which is a pseudocircle. Furthermore, any map which is sufficiently close (in the C^1 -metric) to the constructed map has a similar set.

I. INTRODUCTION

In 1982, Handel [Ha] gave an example of a C^∞ -diffeomorphism of the plane with an extremely pathological strange attractor. The attractor for his diffeomorphism is a plane-separating continuum known to topologists as a pseudocircle. A *continuum* is a compact, connected, metric space. A continuum X is *indecomposable* if it cannot be written as the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable. Pseudocircles are hereditarily indecomposable continua. Since pseudocircles are hereditarily indecomposable, they cannot contain any arcs, that is, continua homeomorphic to the unit interval $[0, 1]$.

Several modifications of Handel's example have appeared since 1982. These include results by Herman [He], Moeckel [M], and Pommerenke and Rodin [PR]. As a result of these investigations, and many others, it appears that even for smooth dynamical systems, invariant sets are sometimes quite irregular.

In this paper we construct an example of a dynamical system which we feel is not pathological. Hence our approach and our goals are quite different from those in the papers mentioned. The example we construct is a map F_0 on a 3-manifold. In the example there are two compact, connected sets R_1 and R_2 which have interior, and a map F_0 , which has the following properties. The map F_0 is continuous in a neighborhood N of $R_1 \cup R_2$, and $F_0(R_i) \supseteq N$ for $i = 1, 2$, and $F_0(\partial R_i)$ is disjoint from $R_1 \cup R_2$ for $i = 1, 2$.

Write $\mathbb{B} = \{q|F_0^n(q) \in R_1 \cup R_2 \text{ for all } n\}$. This set \mathbb{B} has uncountably many components, each of which is nowhere dense. Each component corresponds to a sequence $a = \{a_i\}_0^\infty$, where each $a_i \in \{1, 2\}$. For each such sequence there is a unique component of \mathbb{B} , denoted $B(a) = \{q|F_0^n(q) \in R_{a_n} \text{ for } n = 0, 1, 2, \dots\}$.

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In our case R_1 and R_2 are each the product of a disk and a circle. Each component of B is in some sense circlelike, and uncountably many of these components are pseudocircles. The construction depends only on how the maps are defined on $R_1 \cup R_2$ and does not even depend on the 3-manifold in which we place $R_1 \cup R_2$, but for notational simplicity, we choose a specific 3-manifold and define F_0 on the entire manifold.

In the construction below there is a discontinuity (outside a compact neighborhood of $R_1 \cup R_2$), but this discontinuity can be removed by embedding the compact neighborhood in a different 3-manifold such as \mathbb{R}^3 or S^3 (in which R_1 and R_2 are contractible). Our construction is perturbable in the sense that for any map which is sufficiently close (in the C^1 topology) to F_0 we still have uncountably many components of \mathbb{B} that are pseudocircles.

We construct a map F_0 from $S^1 \times \mathbb{R}^1 \times S^1$ to itself, write elements in that space $q = (x, y, z)$, and refer to the x , y , and z coordinates of q as well as the first, second, and third coordinates of q . The objective is to investigate the dynamical system $q_{n+1} = F_0(q_n)$. The map is chosen so that for every starting point q_0 the trajectory q_n has its y coordinate go either to $+\infty$ or to $-\infty$, or remain bounded as n approaches ∞ , and we refer to initial points q_0 as either $+\infty$ -points, $-\infty$ -points, or bounded points. Our results concern what happens for particular initial values of z . For $z \in S^1$, let $\tilde{B}_z = \{(x, y) | (x, y, z) \text{ is a bounded point}\}$. For uncountably many choices of z , \tilde{B}_z is a pseudocircle in $S^1 \times \mathbb{R}^1$ which separates $S^1 \times \mathbb{R}^1$ into two open sets, each homeomorphic to $S^1 \times \mathbb{R}^1$. For each such z , $\tilde{B}_z \times \{z\}$ is a pseudocircle in the space, and the union of these pseudocircles $\tilde{B}_z \times \{z\}$ forms an invariant set (the invariant set mentioned in the abstract) in the space.

There are both advantages and disadvantages associated with our particular construction. Choose disjoint intervals I_1 and I_2 in S^1 such that (1) $[-\frac{1}{128}, \frac{41}{128}] = I_1$ (so that I_1 is a fattened up $[0, \frac{1}{4}]$), and (2) $[\frac{63}{128}, \frac{109}{128}] = I_2$ (so that I_2 is a fattened up $[\frac{1}{2}, \frac{3}{4}]$). The map F_0 is chosen so that $F_1 = F_0|(S^1 \times \mathbb{R}^1 \times I_1)$ is a C^∞ map, and the map $F_2 = F_0|(S^1 \times \mathbb{R}^1 \times I_2)$ is chosen so that both the map and its inverse are C^∞ maps. However, as long as the space is $S^1 \times \mathbb{R}^1 \times S^1$, F_0 cannot be chosen so that it is continuous and F_1 cannot be chosen so that it is one-to-one. In the previously mentioned investigations, the sets of interest are attractors or minimal invariant sets, and the maps are C^∞ diffeomorphisms. Thus, with regard to these aspects, our example has disadvantages compared to the previous investigations. We hope to change our example by raising the dimension of our space, so that F_0 will be a diffeomorphism on a manifold of dimension 7 and all diffeomorphisms close to F_0 will have the same properties. But that will have to be left for future work.

On the advantage side, we believe our example is (1) natural, in the sense that it could actually model a physical or naturally occurring system; and is (2) perturbable. The constructions of the earlier examples goes something like the following: There is a diffeomorphism h_1 of \mathbb{R}^2 that satisfies properties (1) to (n) . Having h_1 , it is possible to choose a diffeomorphism h_2 so that not only are properties (1) to (n) satisfied, but also h_2 "dovetails" with h_1 . Then h_3 is chosen to "dovetail" with h_1 and h_2 and to satisfy properties (1) to (n) . The process continues, and the result is a diffeomorphism h on \mathbb{R}^2 obtained as a limit of the diffeomorphisms, h_1, h_2, \dots . Thus, h , although nice in

many respects, is somewhat fragile. Such constructions are not perturbable; i.e., diffeomorphisms close to h would admit much simpler (topologically) invariant sets.

Dynamical systems often exhibit expansion or contraction in one or more directions, as well as shearing and rippling behaviors, and attracting or repelling fixed points. Our map is constructed by combining a finite and explicit set of these behaviors. Specifically, two maps, T (for “two”) and W (for “wobble”), are constructed on $S^1 \times \mathbb{R}^1$, and a map g on S^1 is constructed. Two disjoint intervals I_1 and I_2 are selected and the map F_0 on the $S^1 \times \mathbb{R}^1 \times S^1$ is chosen so that $F_0(x, y, z) = (\mu_z(x, y), g(z))$ with $\mu_z = T$ if $z \in I_1$ and $\mu_z = W$ if $z \in I_2$. (We do not specify F_0 outside of $S^1 \times \mathbb{R}^1 \times (I_1 \cup I_2)$.) If $S^1 \times \mathbb{R}^1 \times S^1$ is then embedded in \mathbb{R}^3 or S^3 in a nice, smooth way, it is then possible to extend F_0 defined on the embedded image of $S^1 \times \mathbb{R}^1 \times S^1$ in a smooth way. We denote the extended map on \mathbb{R}^3 or S^3 by F_0 also, as we can depend on context to avoid ambiguity. Then there exists an open set u in the C^1 maps on \mathbb{R}^3 or S^3 such that u contains F_0 and such that if $F \in u$, then F also has an invariant set with an uncountable number of pseudocircle components. In this sense, our dynamical system is perturbable.

II. BACKGROUND AND NOTATION

If (X, d) is a metric space, let $H(X) = \{h|h \text{ is a homeomorphism from } X \text{ onto itself}\}$. $C^0(X) = \{f|f \text{ is a continuous map from } X \text{ to itself}\}$, and, whenever X is a differentiable manifold, $C^1(X) = \{f|f \text{ is a differentiable map from } X \text{ to itself}\}$. For $\varepsilon > 0$, $h \in H(X)$, $f \in C^0(X)$, and $\hat{f} \in C^1(X)$, let

$$\begin{aligned} N_\varepsilon(h) &= \{k \in H(X) | d(k(x), h(x)) < \varepsilon \text{ for } x \in X\}, \\ N_\varepsilon^0(f) &= \{k \in C^0(X) | d(k(x), f(x)) < \varepsilon \text{ for } x \in X\}, \\ N_\varepsilon^1(\hat{f}) &= \{k \in C^1(X) | d(k(x), \hat{f}(x)) < \varepsilon, \\ &\quad \text{and } \|Dk(x) - D\hat{f}(x)\| < \varepsilon, \text{ for each } x \in X\}, \end{aligned}$$

where Dk and $D\hat{f}$ and the Jacobian matrices and $\|\cdot\|$ is a norm on the space of matrices.

For purposes of the proofs the metric spaces that interest us here are S^1 , $S^1 \times \mathbb{R}^1$, $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$, $S^1 \times \mathbb{R}^1 \times S^1$, and $\mathbb{R}^1 \times \mathbb{R}^1 \times S^1$. Since we consider S^1 as $\mathbb{R}^1 \bmod 1$, for $a, b \in S^1$, we take the distance from a to b in S^1 to be $\text{dist}(a, b)$, the length of the shorter arc in S^1 between a and b . For (x, y) and $(x', y') \in \mathbb{R}^2$, the distance from (x, y) to (x', y') is $\sqrt{(x - x')^2 + (y - y')^2}$, and for (x, y) and $(x', y') \in S^1 \times \mathbb{R}^1$, the distance from (x, y) to (x', y') is $\sqrt{(\text{dist}(x, x'))^2 + (y - y')^2}$. Distances in $S^1 \times \mathbb{R}^1 \times S^1$ and $\mathbb{R}^1 \times \mathbb{R}^1 \times S^1$ are defined analogously. In all three cases, d denotes distance, i.e., for a and b points of one of these spaces, $d(a, b)$ denotes the distance from a to b in the space. We rely on the context to avoid ambiguity. Likewise, if $\varepsilon > 0$ and q is a point in one of our spaces, then $D_\varepsilon(q)$ denotes the set of all points of the space at a distance less than ε from q .

We use π_1, π_2 to denote the projections of $S^1 \times \mathbb{R}^1$ or $\mathbb{R}^1 \times \mathbb{R}^1$ to their respective coordinate spaces. When discussing $S^1 \times \mathbb{R}^1 \times S^1$ we speak of π_3 as well. Let \mathbb{N} denote the positive integers and \mathbb{Z} denote the integers.

A finite collection $C = \{c_0, c_1, \dots, c_n\}$ of sets is a circular chain if $c_i \cap c_j \neq$

\emptyset if and only if $|i - j| \leq 1$ or $\{i, j\} = \{0, n\}$. The elements of a circular chain are called *links*. In this paper the circular chains have links that are closures of open sets, and adjacent links intersect only at their boundaries. A continuum is *circularly chainable* or *circlelike* if for each $\varepsilon > 0$, it has a circular chain cover of mesh less than ε . Pseudocircles can be characterized as circularly chainable, hereditarily indecomposable continua which can be embedded in any 2-manifold. A pseudocircle embedded in a cylinder ($S^1 \times \mathbb{R}^1$) separates that cylinder into two open, connected sets. Pseudocircles have been widely studied, and have many startling properties. For most mathematicians these objects are outside the intuition and seem to be without structure. Quite the reverse is true: these objects have a rich structure. For more information and references regarding pseudocircles we refer the reader to [B, F1, F2, R, KR].

Finally, when proving that a continuum X is hereditarily indecomposable, the following fundamental theorem is often useful. It appeared in the form given here in [OT], but it is due to Krasinkiewicz and Minc [Kr, KM].

Krasinkiewicz-Minc Theorem. *Let X be a continuum. Then the following are equivalent:*

- (I) X is hereditarily indecomposable.
- (II) For every pair of disjoint, closed subsets A and B of X and for every open set U intersecting all components of A , there exist closed subsets M and N of X such that
 - (1) $X = M \cup N$,
 - (2) $A \subseteq M$, $B \subseteq N$, and
 - (3) $M \cap N \subseteq U - A \cup B$.
- (III) For every pair of disjoint, closed subsets A and B of X and for every pair of neighborhoods U of A and V of B , there exist closed subsets X_0 , X_1 , and X_2 of X such that
 - (1) $X = X_0 \cup X_1 \cup X_2$,
 - (2) $A \subseteq X_0$ and $B \subseteq X_2$,
 - (3) $X_0 \cap X_2 = \emptyset$, and
 - (4) $X_0 \cap X_1 \subseteq V$ and $X_1 \cap X_2 \subseteq U$.

III. RESULTS AND PROOFS

First, we define maps α , σ , s , and T on $S^1 \times \mathbb{R}^1$ and $\mathbb{R}^1 \times \mathbb{R}^1$. (We rely on context to tell whether the space involved is $S^1 \times \mathbb{R}^1$ or $\mathbb{R}^1 \times \mathbb{R}^1$. The maps defined make sense in both spaces, the difference being that in $S^1 \times \mathbb{R}^1$, $(x, y) = (x + 1, y)$. That is, we think of S^1 as the quotient space \mathbb{R}^1/\mathcal{Z} .)

For $(x, y) \in S^1 \times \mathbb{R}^1$ or $\mathbb{R}^1 \times \mathbb{R}^1$, define $\sigma(x, y) = (x - y, y)$, $s(x, y) = (x, 128y - 127 \sin 2\pi x) = (x, y + 127(y - \sin 2\pi x))$, and $T(x, y) = (2x, 2y)$. Note that σ and s are homeomorphisms in both spaces with $\sigma^{-1}(x, y) = (x + y, y)$ and $s^{-1}(x, y) = (x, \frac{1}{128}y + \frac{127}{128} \sin 2\pi x)$. The map T is a homeomorphism on $\mathbb{R}^1 \times \mathbb{R}^1$ but not on $S^1 \times \mathbb{R}^1$, with $T^{-1}(x, y) = (\frac{1}{2}x, \frac{1}{2}y)$ for $(x, y) \in \mathbb{R}^1 \times \mathbb{R}^1$. Also, $T: S^1 \times \mathbb{R}^1 \rightarrow S^1 \times \mathbb{R}^1$ is *not* homotopic to the identity. However, the lift $T: \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^1 \times \mathbb{R}^1$ is a diffeomorphism homotopic to the identity.

Let $B_0 = S^1 \times \{0\}$, or, depending on context to avoid ambiguity, $\mathbb{R}^1 \times \{0\}$. Likewise, let $A_0 = S^1 \times [-2, 2]$ or $\mathbb{R}^1 \times [-2, 2]$. Let $R = \{(x, y) | y = \sin 2\pi x\}$.

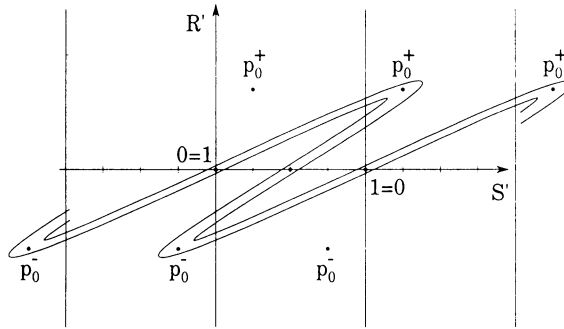


FIGURE 1

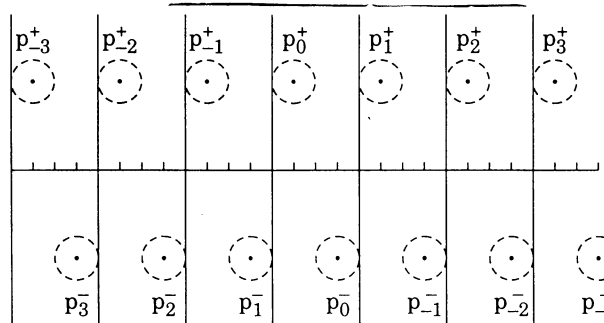


FIGURE 2

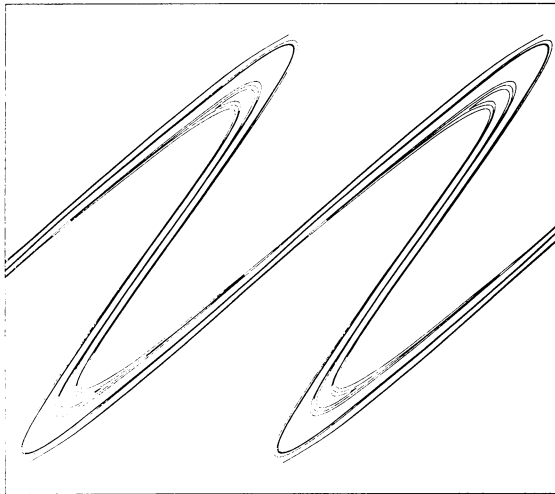
Thus, R is the graph of the sine function with $R \subseteq \mathbb{R}^1 \times \mathbb{R}^1$ and $R \subseteq S^1 \times \mathbb{R}^1$ (depending on context). Note that each point of R is fixed under s . See Figure 1.

Let $p_0^+ = (\frac{1}{4}, 1)$ and $p_0^- = (\frac{3}{4}, -1)$. For $n \in \mathcal{Z}$, $p_n^+ = (n + \frac{1}{4}, 1)$ and $p_n^- = (\frac{3}{4} - n, -1)$. (In $S^1 \times \mathbb{R}^1$, $p_n^+ = p_0^+$ for all n and $p_n^- = p_0^-$ for all n . See Figure 2. Note that $\sigma^{-1}(p_i^\pm) = p_{i+1}^\pm$.)

For $(x, y) \notin D_{1/4}(p_n^+) \cup D_{1/4}(p_n^-)$ for any n , $\alpha(x, y) = (x, y)$. For $n \in \mathcal{Z}$, $p = p_n^+$ or p_n^- , each point (x, y) in $\overline{D_{1/4}(p)} - \{p\}$ is on exactly one radial line L through p in $\overline{D_{1/4}(p)}$. Define α so that $\alpha(L) = L$ with $\alpha(p) = p$. Further, if $(x, y) \in \overline{D_{3/16}(p)}$, then $d_{(x,y)} = d(p, (x, y)) \leq \frac{3}{16}$. Define $\alpha(x, y)$ to be that unique point on L at a distance of $d_{(x,y)}^4 + 2^{-7}d_{(x,y)} = d_{(x,y)} + (d_{(x,y)}^3 + 2^{-7})$. For (x, y) outside $\overline{D_{3/16}(p)}$ but in $D_{1/4}(p)$, define $\alpha(x, y)$ in such a way that $\alpha(x, y)$ is a point on L closer to p than is (x, y) and in such a way that the resulting α is a diffeomorphism (on both $\mathbb{R}^1 \times \mathbb{R}^1$ and $S^1 \times \mathbb{R}^1$). Let $W = \alpha\sigma^{-1}s^{-1}$. Then $W^{-1} = s\sigma\alpha^{-1}$. See Figure 3.

For a finite collection f_1, f_2, \dots, f_n of maps on a space, let $\hat{f}_n = f_n \circ f_{n-1} \circ \dots \circ f_1$.

Lemmas 1–4 and Theorem 5 prove that (1) it is possible to choose a sequence g_1, g_2, \dots of the maps T and W^{-1} so that $\bigcap_{n=1}^\infty (\hat{g}_n)^{-1}(A_0)$ is a pseudocircle, and (2) for each choice of sequence g_1, g_2, \dots of maps T and W^{-1} , $\bigcap_{n=1}^\infty (\hat{g}_n)^{-1}(A_0)$ is a continuum that separates $S^1 \times \mathbb{R}^1$ into two disjoint, open sets each homeomorphic to $S^1 \times \mathbb{R}^1$. Further, they show that even if the respective sequences consist of maps close to the maps T and W^{-1} , then the

FIGURE 3. Computer generated approximation of $W^3(A_0)$

results hold. In order to prove our final result, it is necessary to extend these results to maps on $S^1 \times \mathbb{R}^1 \times S^1$, but the lemmas, theorem, and proofs given here generalize easily to that situation, which is discussed later.

It may help, as far as understanding the arguments here, to remember that the map W gives the indecomposability present in the invariant set, and that Lemmas 4 and 4' make this precise. Lemmas 1 and 2 work together to give Lemma 4 and then Lemma 4'. However, the indecomposability given the invariant set by W is only an indecomposability "in the large." To give the hereditary indecomposability it is necessary to adjust the sizes of the "zigzags" associated with W so that all sizes are present, and that is the role of T in the construction. Lemma 3 makes the properties of T precise.

Lemma 1 (Space: $S^1 \times \mathbb{R}^1$). *Let ε be a positive number less than 2^{-7} . Suppose that for each $i \in \{1, \dots, n\}$, $W_i \in N_{\varepsilon/\pi}(W)$. Then if $p = p_0^+$ or p_0^- , and $q \in D_{1/16\pi+\varepsilon/\pi}(p)$, then $\widehat{W}_n(q) \in \overline{D_{1/16\pi+\varepsilon/\pi}(p)}$.*

Proof. For $\beta < \frac{1}{4}$, $(1 - \cos(2\pi\beta))^2 + (\sin(2\pi\beta))^2 < 2\pi\beta$. (Consider the unit circle.) This leads to the inequality

$$1 - \pi\beta < \cos(2\pi\beta) = \sin(\pi/2 \pm 2\pi\beta) = \sin 2\pi(\frac{1}{4} \pm \beta).$$

Thus, if

$$(a, b) \in D_\beta(p_0^+) \subseteq [\frac{1}{4} - \beta, \frac{1}{4} + \beta] \times [1 - \pi\beta, 1 + \pi\beta],$$

then

$$s^{-1}(a, b) \in [\frac{1}{4} - \beta, \frac{1}{4} + \beta] \times [1 - \pi\beta, 1 + \pi\beta],$$

because s^{-1} takes vertical lines to vertical lines. Similarly, $\sin 2\pi(-\frac{1}{4} \pm \beta) < -1 + \pi\beta$. Let $p = (p_1, p_2)$ and

$$B = [p_1 - \frac{1}{16\pi} - \frac{\varepsilon}{\pi}, p_1 + \frac{1}{16\pi} + \frac{\varepsilon}{\pi}] \times [p_2 - \frac{1}{16} - \varepsilon, p_2 + \frac{1}{16} + \varepsilon].$$

Since $s^{-1}B \subseteq B$, $s^{-1}(q) \in B$. Let $s^{-1}(q) = (q_1, q_2)$. If $p = (\frac{1}{4}, 1)$,

$$d(\sigma^{-1}s^{-1}(q), p) = ((q_1 + q_2 - 1 - \frac{1}{4})^2 + (q_2 - 1)^2)^{1/2} \leq (5(\frac{1}{16} + \varepsilon)^2)^{1/2} < \frac{3}{16}.$$

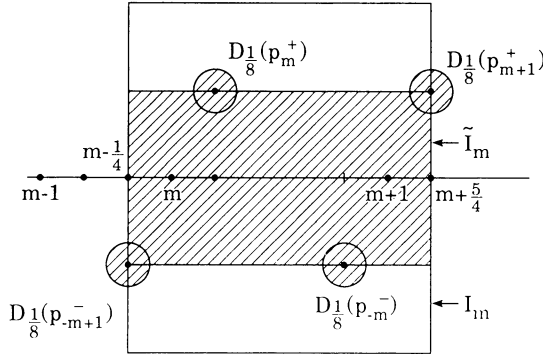


FIGURE 4

If $p = (\frac{3}{4}, -1)$, a similar argument yields that $d(\sigma^{-1}s^{-1}(q), p) < \frac{3}{16}$. Then $W(q) = \alpha\sigma^{-1}s^{-1}(q) \in D_{1/64}(p)$ and $W_1(q) \in D_{1/16\pi+\varepsilon/\pi}(p)$. Thus, we can continue: since $W_1(q) \in D_{1/16\pi+\varepsilon/\pi}(p)$, $WW_1(q) \in D_{1/64}(p)$, and $\widehat{W}_2(q) \in D_{1/16\pi+\varepsilon/\pi}(p)$. Eventually we obtain the result desired. \square

For each $m \in \mathcal{Z}$, let

$$I_m = A_0 \cap ([m - \frac{1}{4}, m + \frac{5}{4}] \times \mathbb{R}^1)$$

and let

$$\tilde{I}_m = ([m - \frac{1}{4}, m + \frac{5}{4}] \times [-1, 1]) \cup D_{1/8}(p_m^+) \cup D_{1/8}(p_{m+1}^+) \cup D_{1/8}(p_{-m}^-) \cup D_{1/8}(p_{-m+1}^-).$$

See Figure 4.

Lemma 2 (Space: \mathbb{R}^2). Suppose $0 < \varepsilon < 2^{-7}/\pi$. For each $m \in \mathcal{Z}$, $n \in \mathbb{N} \cup \{0\}$, $W[D_\varepsilon(\bigcup_{i=-n}^n \tilde{I}_{m+i})] \subseteq \bigcup_{i=-n-1}^{n+1} \tilde{I}_{m+i}$. If for every $i \leq n$, $W_i \in N_\varepsilon(W)$, then $\widehat{W}_n \tilde{I}_m \subseteq D_\varepsilon(\bigcup_{i=-n}^n \tilde{I}_{m+i})$.

Proof. For $n' \in \mathbb{N} \cup \{0\}$, let $\theta_{n'} = [m - \frac{1}{4} - n', m + \frac{5}{4} + n'] \times [-1, 1]$ and let

$$\Gamma_{n'} = \left(\bigcup_{i=-n'}^{n'+1} D_{1/8}(p_{m+i}^+) \right) \cup \left(\bigcup_{i=-n'}^{n'+1} D_{1/8}(p_{-m+i}^-) \right).$$

Note that $\bigcup_{i=-n'}^{n'+1} \tilde{I}_{m+i} = \theta_{n'} \cup \Gamma_{n'}$.

Without loss of generality, suppose $m = 0$. Suppose then that $(x, y) \in D_\varepsilon(\bigcup_{i=-n}^n \tilde{I}_i)$. If $W(x, y) \in \theta_{n+1}$, we are done, for $\theta_{n+1} \subseteq \bigcup_{i=-n-1}^{n+1} \tilde{I}_i$. If $W(x, y) \notin \theta_{n+1}$, then $\sigma^{-1}s^{-1}(x, y) = (\hat{x}, \hat{y}) \notin \theta_{n+1}$, for otherwise $(\hat{x}, \hat{y}) \in \theta_{n+1}$ and $\alpha(\theta_{n+1}) = \theta_{n+1}$. (Recall that α either does not move (\hat{x}, \hat{y}) , or it moves it along a radial line L containing some p_{m+i}^+ or p_{-m+i}^- and contained in θ_{n+1} .)

Now $\sigma^{-1}s^{-1}(x, y) = (\hat{x}, \hat{y}) \notin \theta_{n+1}$ implies that $\hat{y} = \frac{1}{128}y + \frac{127}{128}\sin 2\pi x > 1$ or $\hat{y} < -1$, or $\hat{x} = x + \frac{1}{128}y + \frac{127}{128}\sin 2\pi x > n + \frac{9}{4}$ or $\hat{x} < -n - \frac{5}{4}$. If $\hat{x} > n + \frac{9}{4}$, then $\hat{y} > \frac{7}{8} - \varepsilon$ (because $\hat{x} = x + \hat{y}$ and $x \leq n + \frac{5}{4} + \frac{1}{8} + \varepsilon$). Likewise, if $\hat{x} < -n - \frac{5}{4}$, then $\hat{y} < -\frac{7}{8} + \varepsilon$. Further, x must be within $\frac{1}{8}$ unit from some $\pm \frac{1}{4} + j$, for otherwise $-\sqrt{2}/2 \leq \sin 2\pi x \leq \sqrt{2}/2$ and $-\frac{7}{8} + \varepsilon < .83 < \hat{y} = \frac{1}{128}y + \frac{127}{128}\sin 2\pi x < .83 < \frac{7}{8} - \varepsilon$, which is impossible. If $\frac{7}{8} - \varepsilon < \hat{y} \leq 1$, then

$\hat{x} = x + \hat{y} < n + \frac{9}{4} + \frac{1}{8}$, and $(\hat{x}, \hat{y}) \in \bigcup_{i=-n-1}^{n+1} \tilde{I}_i$. Likewise, if $-1 \leq \hat{y} < -\frac{7}{8} + \varepsilon$, then $\hat{x} = x + \hat{y} > -n - \frac{1}{4} - 1 - \frac{1}{8} = -n - \frac{5}{4} - \frac{1}{8}$, and $(\hat{x}, \hat{y}) \in \bigcup_{i=-n-1}^{n+1} \tilde{I}_i$. Suppose then that $\sin 2\pi x > \sqrt{2}/2$ and $\hat{y} > 1$, or $\sin 2\pi x < -\sqrt{2}/2$ and $\hat{y} < -1$. Suppose first that $\sin 2\pi x > \sqrt{2}/2$ and $\hat{y} > 1$. For some p_i^+ , $|x - \pi_1 p_i^+| < \frac{1}{8}$. Then

$$\begin{aligned} & d^2(\sigma^{-1}s^{-1}(x, y), p_{i+1}^+) \\ &= \left(x + \frac{1}{128}y + \frac{127}{128} \sin 2\pi x - \left(i + \frac{5}{4} \right) \right)^2 \\ &\quad + \left(\frac{1}{128}y + \frac{127}{128} \sin 2\pi x - 1 \right)^2 \\ &\leq \left(x - \left(i + \frac{1}{4} \right) \right)^2 + 2 \left| x - \left(i + \frac{1}{4} \right) \right| \left| \frac{1}{128}y + \frac{127}{128} \sin 2\pi x - 1 \right| \\ &\quad + 2 \left(\frac{1}{128}y + \frac{127}{128} \sin 2\pi x - 1 \right)^2 \\ &< \left(\frac{1}{8} \right)^2 + 2 \left(\frac{1}{8} \right) \left(\frac{1}{128} + \frac{\varepsilon}{128} \right) + 2 \left(\frac{1}{128} + \frac{\varepsilon}{128} \right)^2 \\ &< \frac{1}{64} + \frac{1}{4} \left(\frac{1}{128} + \frac{1}{2^{14}} \right) + 2 \left(\frac{1}{128} + \frac{1}{2^{14}} \right) \\ &= \frac{1}{64} + \left(\frac{1}{128} + 2^{-14} \right) \left(\frac{9}{4} \right) < \left(\frac{3}{16} \right)^2. \end{aligned}$$

Thus, $d(\sigma^{-1}s^{-1}(x, y), p_{i+1}^+) < \frac{3}{16}$, and $d(W(x, y), p_{i+1}^+) < \frac{3}{16}((\frac{3}{16})^3 + 2^{-7}) < .05$. Then $W(x, y) \in \Gamma_{n+1}$. If $\sin 2\pi x < -\sqrt{2}/2$ and $\hat{y} < -1$, a similar proof yields the result $W(x, y) \in \Gamma_{n+1}$. Thus, we have proved the first statement.

Since $W(\tilde{I}_0) \subseteq \bigcup_{i=-1}^1 \tilde{I}_i$, $W_1(\tilde{I}_0) \subseteq D_\varepsilon(\bigcup_{i=-1}^1 \tilde{I}_i)$ and

$$WW_1(\tilde{I}_0) \subseteq W \left(D_\varepsilon \left(\bigcup_{i=-1}^1 \tilde{I}_i \right) \right) \subseteq \bigcup_{i=-2}^2 \tilde{I}_i.$$

Then $\widehat{W}_2(\tilde{I}_0) \subseteq D_\varepsilon(\bigcup_{i=-2}^2 \tilde{I}_i)_1$ so that $W\widehat{W}_2(\tilde{I}_0) \subseteq \bigcup_{i=-3}^3 \tilde{I}_i$. Continuing this process, we obtain the result. \square

Lemma 3 (Space: $S^1 \times \mathbb{R}^1$). Suppose that $0 < \varepsilon < 2^{-7}/\pi$, m is a positive number, and $\tilde{T} \in C^1(S^1 \times \mathbb{R}^1)$ with $\tilde{T} \in N_\varepsilon^1(T)$. If $A \subseteq S^1 \times \mathbb{R}^1$, $\frac{1}{2} > \text{diam } \pi_1 A$, and $\text{diam } A > m$, then $\text{diam } \tilde{T}(A) > \sqrt{3}m$. If $A = [a, b] \times [c, d] \subseteq S^1 \times \mathbb{R}^1$ with $b - a < 1$ and $\text{diam } A < m$, then $\tilde{T}^{-1}(A)$ has two components A_1 and A_2 , each of which maps onto A , and $\text{diam } A_i < m/\sqrt{3}$ for $i = 1, 2$.

Proof. This follows from standard arguments and is omitted, although we remark that it is here that the differentiability of \tilde{T} is important. (If $\tilde{T} \in N_\varepsilon^0(T) - N_\varepsilon^1(T)$, then it is possible for $\tilde{T}^{-1}(A)$ (for $A = [a, b] \times [c, d]$, $b - a < 1$) to have more than two components. Further, if $\text{diam } A < m < \varepsilon$, it is possible that $\text{diam } \tilde{A} \geq m/\sqrt{3}$ for some component \tilde{A} of $\tilde{T}^{-1}(A)$.) \square

Lemma 4. One-component, three-component property (Space: \mathbb{R}^2). Suppose $0 < \varepsilon < 1/128\pi$, $k \in \mathbb{N}$, $n \in \mathcal{Z}$, and $\frac{3}{2} > \delta > \frac{9}{8} + \varepsilon$. If for each $i \in \{1, \dots, k\}$, $W_i \in N_\varepsilon^1(W)$, then $\widehat{W}_k(A_0)$ has the following properties:

- (1) *One-component property.* $\widehat{W}_k(A_0) \cap ([\pi_1 p_{-n+k+1}^-, \pi_1 p_{n+k+1}^+] \times \mathbb{R}^1)$ has exactly one component E that intersects both $(D_{1/16\pi+\varepsilon}(\pi_1 p_{n+k+1}^+) \times \mathbb{R}^1)$ and $(D_{1/16\pi+\varepsilon}(\pi_1 p_{-n+k+1}^-) \times \mathbb{R}^1)$.
- (2) *Three-component property.* $\widehat{W}_k(A_0) \cap ([\pi_1 p_{-n+k+1}^- + \delta, \pi_1 p_{n+k+1}^+ - \delta] \times \mathbb{R}^1)$ and $E \cap ([\pi_1 p_{-n+k+1}^- + \delta, \pi_1 p_{n+k+1}^+ - \delta] \times \mathbb{R}^1)$ each have exactly three components that intersect both $\{\pi_1 p_{-n+k+1}^- + \delta\} \times \mathbb{R}^1$ and $\{\pi_1 p_{n+k+1}^+ - \delta\} \times \mathbb{R}^1$.

Proof. We proceed by induction. Let $k = 1$. Fix n . If L is a horizontal interval at height between -2 and 2 and extending from $\pi_1 p_{-n+1}^-$ to $\pi_1 p_{n+1}^+$, then $W(L)$ is a path beginning within $\frac{1}{64}$ of the point p_{-n+2}^- ; then increasing from there to within $\frac{1}{64}$ of p_{n+2}^+ ; then decreasing, with respect to both vertical and horizontal directions, to within $\frac{1}{64}$ of p_{-n+1}^- ; and finally increasing, with respect to both directions, to within $\frac{1}{64}$ of p_{n+2}^+ . Since $W_1 \in N_\varepsilon^1(W)$, $W_1(L)$ is within $\frac{1}{128\pi}$ of $W(L)$. The path $W_1(L)$ does not extend beyond $1/64 + 1/128\pi$ of any point on the vertical line through $(n + \frac{9}{4}, 1)$ nor does it extend beyond $1/64 + 1/128\pi$ of any point on the vertical line through $(n - \frac{5}{4}, -1)$, and it only comes close to the points $(n + \frac{9}{4}, 1)$ and $(n - \frac{5}{4}, -1)$ on these respective vertical lines. (Lemmas 1 and 2 guarantee these results. See Figure 5.)

For $n \in \mathbb{Z}$, let $A_0^n = [n - \frac{1}{4}, n + \frac{5}{4}] \times [-2, 2]$. Then $W_1(A_0^n)$ consists of a continuous collection of paths beginning close to p_{-n+2}^- , increasing almost to p_{n+1}^+ , decreasing almost to p_{-n+1}^- and increasing almost to p_{n+2}^+ , and because of Lemmas 1 and 2, the result follows for $W_1(A_0^n)$. The result then follows easily for $W_1(A_0) = W_1(\bigcup_{n \in \mathcal{Z}} A_0^n)$.

Suppose that the lemma holds for $k - 1$ and each $n \in \mathcal{Z}$. Then

- (1) $\widehat{W}_{k-1}(A_0) \cap ([\pi_1 p_{-n+k}^-, \pi_1 p_{n+k}^+] \times \mathbb{R}^1)$ has exactly one component E_{k-1} that intersects both $(D_{1/16\pi+\varepsilon}(\pi_1 p_{n+k}^+) \times \mathbb{R}^1)$ and $(D_{1/16\pi+\varepsilon}(\pi_1 p_{-n+k}^-) \times \mathbb{R}^1)$, and
- (2) $\widehat{W}_{k-1}(A_0) \cap ([\pi_1 p_{-n+k}^- + \delta, \pi_1 p_{n+k}^+ - \delta] \times \mathbb{R}^1)$ and $E_{k-1} \cap ([\pi_1 p_{-n+k}^- + \delta, \pi_1 p_{n+k}^+ - \delta] \times \mathbb{R}^1)$ each contain exactly three components that intersect both $\{\pi_1 p_{-n+k}^- + \delta\} \times \mathbb{R}^1$ and $\{\pi_1 p_{n+k}^+ - \delta\} \times \mathbb{R}^1$.

Consider $\widehat{W}_k(A_0) = W_k(\widehat{W}_{k-1}(A_0))$. Fix n . If L is a horizontal interval at height between -2 and 2 and extending from within $1/16\pi + \varepsilon$ of $\pi_1 p_{-n+k}^-$ to within $1/16\pi + \varepsilon$ of $\pi_1 p_{n+k}^+$, then $W_k(L)$ is a path whose left most endpoint is within $1/16\pi + \varepsilon$ of p_{-n+k+1}^- and whose rightmost endpoint is within $1/16\pi + \varepsilon$ of p_{n+k+1}^+ . From its left endpoint, the path $W_k(L)$ increases (with respect to both the x and y directions) to within $1/16\pi + \varepsilon$ of p_{-n+k+2}^+ ; decreases (both x and y directions) to within $1/16\pi + \varepsilon$ of p_{-n+k}^- ; increases to within $1/16\pi + \varepsilon$ of p_{n-k+3}^+ ; ...; decreases from within $1/16\pi + \varepsilon$ of p_{n+k}^+ to within $1/16\pi + \varepsilon$ of p_{-n-k-2}^- ; and finally increases from within $1/16\pi + \varepsilon$ of p_{-n-k+2}^- to within $1/16\pi + \varepsilon$ of p_{n+k+1}^+ . Then $W_k([\pi_1 p_{-n+k}^-, \pi_1 p_{n+k}^+] \times [-2, 2])$ is a continuous collection of paths increasing and decreasing in the same manner from within

$1/16\pi + \varepsilon$ of p_{-n+k+1}^- to within $1/16\pi + \varepsilon$ of p_{n+k+1}^+ , because of Lemmas 1 and 2. Also because of Lemmas 1 and 2, $\widehat{W}_k(A_0) \cap ([\pi_1 p_{-n+k+1}^-, \pi_1 p_{n+k+1}^+] \times \mathbb{R}^1) \subseteq W_k([\pi_1 p_{-n+k}^-, \pi_1 p_{n+k}^+] \times [-2, 2])$ extends from within $1/16\pi + \varepsilon$ of p_{-n+k+1}^- to within $1/16\pi + \varepsilon$ of p_{n+k+1}^+ , because E_{k-1} , the “main” component of $\widehat{W}_{k-1}(A_0)$ extends from within $1/16\pi + \varepsilon$ of p_{-n+k}^- to within $1/16\pi + \varepsilon$ of p_{n+k}^+ . Further, if the three components of $E_{k-1} \cap ([\pi_1 p_{-n+k}^- + \delta, \pi_1 p_{n+k}^+ - \delta] \times \mathbb{R}^1)$ that intersect both vertical lines $x = \pi_1 p_{-n+k}^- + \delta$ and $x = \pi_1 p_{n+k}^+ - \delta$ are denoted $E_{k-1}^1, E_{k-1}^2, E_{k-1}^3$, then

$$\begin{aligned} W_k(E_{k-1}) \cap ([\pi_1 p_{-n+k+1}^- + \delta, \pi_1 p_{n+k+1}^+ - \delta] \times \mathbb{R}^1) \\ \supseteq [W_k(E_{k-1}^1) \cup W_k(E_{k-1}^2) \cup W_k(E_{k-1}^3)] \\ \cap ([\pi_1 p_{-n+k+1}^- + \delta, \pi_1 p_{n+k+1}^+ - \delta] \times \mathbb{R}^1), \end{aligned}$$

and each $E_k^i = W_k(E_{k-1}^i) \cap ([\pi_1 p_{-n+k+1}^- + \delta, \pi_1 p_{n+k+1}^+ - \delta] \times \mathbb{R}^1)$ is a continuum intersecting both vertical lines $x = \pi_1 p_{-n+k+1}^- + \delta$ and $x = \pi_1 p_{n+k+1}^+ - \delta$. Thus, E_k^1, E_k^2 , and E_k^3 are the three components of the Lemma. See Figure 5. \square

Lemma 4'. **One-component, three-or-more-component property** (Space: \mathbb{R}^2). Suppose $0 < \varepsilon < 1/128\pi$, $k \in \mathbb{N}$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$, and $m \leq k+1$, and $\frac{3}{2} > \delta > \frac{9}{8} + \varepsilon$. If for each $i \in \{1, \dots, k\}$, $W_i \in N_\varepsilon^1(W)$, then $\widehat{W}_k(A_0) \cap ([\pi_1 p_{-n+m}^-, \pi_1 p_{n+m}^+] \times \mathbb{R}^1) = D$ has the following properties:

- (1) *One-component property.* D has a least one component E that intersects both $(D_{1/16\pi+\varepsilon}(\pi_1 p_{n+m}^+) \times \mathbb{R}^1)$ and $(D_{1/16\pi+\varepsilon}(\pi_1 p_{-n+m}^-) \times \mathbb{R}^1)$.
- (2) *Three-component (or more) property.* $E \cap ([\pi_1 p_{-n+m}^- + \delta, \pi_1 p_{n+m}^+ - \delta] \times \mathbb{R}^1)$, where E is a component as in (1), contains at least three components intersecting both the vertical lines through $\pi_1 p_{-n+m}^- + \delta$ and $\pi_1 p_{n+m}^+ - \delta$.

Proof. This lemma is a slight generalization of Lemma 4, and its proof is similar. It is therefore omitted. \square

Theorem 5. *There is a sequence g_1, g_2, \dots of maps on $S^1 \times \mathbb{R}^1$ such that*

- (1) *for each i , $g_i \in N_\varepsilon^1(W^{-1}) \cup N_\varepsilon^1(T)$, and*
- (2) *$\bigcap_{i=0}^\infty (\widehat{g}_i^{-1})(A_0)$ is pseudocircle that separates $S^1 \times \mathbb{R}^1$ into two open sets, each homeomorphic to $S^1 \times \mathbb{R}^1$.*

Proof. Choose $m_1 \in \mathbb{N}$ so that if $\{T_{11}, T_{12}, \dots, T_{1, m_1}\}$ is contained in $N_\varepsilon^1(T)$, then each component of $\widehat{T_{1, m_1}^{-1}}((-\frac{1}{4}, \frac{3}{4}) \times [-2, 2])$ had diameter less than $\frac{1}{2}$. Choose an m_1 -element subset $\{T_{1,1}, \dots, T_{1, m_1}\}$ of $N_\varepsilon^1(T)$. The collection of closures of components $(\widehat{T_{1, m_1}^{-1}}((-\frac{1}{4}, \frac{3}{4}) \times [-2, 2]))$ forms a circular chain cover $C_1 = \{c(1, 1), \dots, c(1, p_1)\}$ of $(\widehat{T_{1, m_1}^{-1}}(A_0))$ of mesh less than $\frac{1}{2}$. Let $F_1 = T_{11} \circ \dots \circ T_{1, m_1}$. For each j , $1 \leq j \leq p_1$, $F_1(c(1, j)) = A_0 = [-\frac{1}{4}, \frac{3}{4}] \times [-2, 2]$.

Now switch the setting from $\mathbb{R}^1 \times S^1$ to \mathbb{R}^2 , i.e., go to the lift of F_1 and covering space of $\mathbb{R}^1 \times S^1$. In this setting, we can think of $F_1(c(1, j))$ as being $[-\frac{1}{4} + j - 1, \frac{3}{4} + j - 1] \times [-2, 2]$. If $p_1 \leq 5$, let $n_1 = 0$. Otherwise

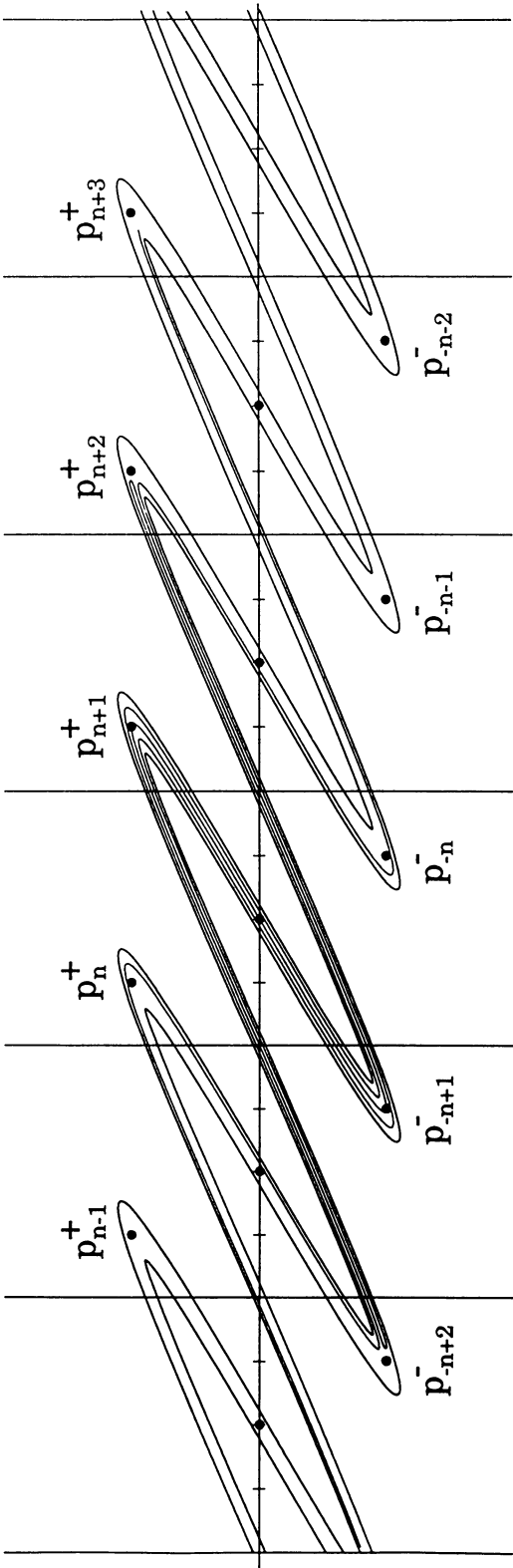


FIGURE 5. $W^2(A_0)$ and $W(A_0)$

choose n_1 to be an integer such that $n_1 > p_1/2$. Choose an n_1 -element subset $\{W_{1,1}, \dots, W_{1,n_1}\}$ of $N_\varepsilon^1(W)$.

Switch the setting back to $\mathbb{R}^1 \times S^1$. Let $A_1 = (\widehat{T_{1,m_1}^{-1}}) \circ \widehat{W_{1,n_1}}(A_0)$. Then $A_1 \subseteq A_0$ and A_1 is covered by the circular chain C_1 . (Recall that with respect to \mathbb{R}^2 and the lift of F_1 , also referred to as F_1 , $F_1(c(1, j)) = [-\frac{1}{4} + j - 1, \frac{3}{4} + j - 1] \times [-2, 2]$ for $1 \leq j \leq p_1$. Choosing $n_1 > p_1/2$ ensures that $(\widehat{W_{1,n_1}})^{-1}([-\frac{1}{4}, p_1 - \frac{1}{4}] \times [-2, 2]) = D_1$ has the one-component—three- (or more) component property of Lemma 4' with respect to $D_0 = [-\frac{1}{4}, p_1 - \frac{1}{4}] \times [-2, 2]$. That is, there is at least one component of $D_0 \cap D_1$ that comes within $1/16\pi + \varepsilon$ of the vertical lines through $-\frac{1}{4}$ and $p_1 - \frac{1}{4}$, and if $\frac{3}{2} > \delta > \frac{9}{8} + \varepsilon$ and E is such a component then $E \cap ([-\frac{1}{4} + \delta, p_1 - \frac{1}{4} - \delta] \times [-2, 2])$ contains at least three components that intersect both the vertical lines $x = -\frac{1}{4} + \delta$ and $x = p_1 - \frac{1}{4} - \delta$. Thus, we have the one-component—three- (or more) component property for D_1 in D_0 and for A_1 in A_0 .) Let $G_1 = W_{1,1}^{-1} \circ \dots \circ W_{1,n_1}^{-1} \circ T_{1,1} \circ \dots \circ T_{1,m_1}$. Choose $m_2 \in \mathbb{N}$ so that if $\{T_{2,1}, \dots, T_{2,m_2}\} \subseteq N_\varepsilon^1(T)$, then each component of $(\widehat{T_{1,m_1}^{-1}}) \circ \widehat{W_{1,n_1}} \circ (\widehat{T_{2,m_2}^{-1}})((-\frac{1}{4}, \frac{3}{4}) \times [-2, 2])$ has diameter less than $\frac{1}{4}$. The collection of closures of components of $(\widehat{T_{1,m_1}^{-1}}) \circ \widehat{W_{1,n_1}} \circ (\widehat{T_{2,m_2}^{-1}})((-\frac{1}{4}, \frac{3}{4}) \times [-2, 2])$ forms a circular chain cover $C_2 = \{c(2, 1), \dots, c(2, p_2)\}$ of $\widehat{A_2} = (\widehat{T_{1,m_1}^{-1}}) \circ (\widehat{W_{1,n_1}}) \circ (\widehat{T_{2,m_2}^{-1}})(A_0)$ of mesh less than $\frac{1}{4}$. Let $F_2 = T_{2,1} \circ \dots \circ T_{2,m_2} \circ W_{1,1}^{-1} \circ \dots \circ W_{1,n_1}^{-1} \circ T_{1,1} \circ \dots \circ T_{1,m_1}$. For each j , $1 \leq j \leq p_2$, $F_2(c(2, j)) = A_0$. Choose n_2 to be an integer $> p_2/2$. (Again, this is to satisfy Lemma 4' for A_2 relative to $\widehat{A_2}$.) Choose an n_2 -element subset $\{W_{2,1}, \dots, W_{2,n_2}\}$ of $N_\varepsilon(W)$. Let $A_2 = (\widehat{T_{1,m_1}^{-1}}) \circ (\widehat{W_{1,n_1}}) \circ (\widehat{T_{2,m_2}^{-1}}) \circ (\widehat{W_{2,n_2}})(A_0)$.

Continue this process. At the i th level, $A_i = (\widehat{T_{1,m_1}^{-1}}) \circ \widehat{W_{1,n_1}} \circ (\widehat{T_{2,m_2}^{-1}}) \circ \dots \circ (\widehat{T_{i,m_i}^{-1}}) \circ \widehat{W_{i,n_i}}(A_0) \subset A_{i-1}$, with A_i covered by the circular chain cover $C_i = \{c(i, 1), \dots, c(i, p_i)\}$, the collection of closures of components of $\widehat{A_i} = (\widehat{T_{1,m_1}^{-1}}) \circ (\widehat{W_{1,n_1}}) \circ \dots \circ (\widehat{T_{i,m_i}^{-1}})(A_0)$, and m_i chosen so that mesh $C_i < 2^{-i}$. Then choose n_i to be an integer $> p_i/2$, and choose $\{W_{i,1}, \dots, W_{i,n_i}\}$ to be an n_i -element subset of $N_\varepsilon^1(W)$.

Note that $A_0 \supseteq A_1 \supseteq \dots$ and $X = \bigcap_{i=0}^\infty A_i$ is a continuum that separates $S^1 \times \mathbb{R}^1$ into two open sets, each homeomorphic to $S^1 \times \mathbb{R}^1$. Since for each i , C_i is a circular chain that covers X and has mesh less than $1/2^i$, X is circlelike.

We need to show that X is hereditarily indecomposable. To this end, suppose that H and K are disjoint, closed subsets of X and that U and V are open sets in $S^1 \times \mathbb{R}^1$ such that $H \subseteq U$, $K \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$. There is some i such that $d(H, X - U) > 2^{4-i}$, $d(K, X - V) > 2^{4-i}$, and $d(\overline{U}, \overline{V}) > 2^{4-i}$.

Consider C_i . Construct the sets X_0 , X_1 , and X_2 as follows:

- (1) If $c(i, j) \cap H \neq \emptyset$, then $c(i, j) \subseteq X_0$.
- (2) If $c(i, j) \cap K \neq \emptyset$, then $c(i, j) \subseteq X_2$.
- (3) If C' is a subchain of C_i such that both end links of C' intersect H , but no link of C' intersects K , then $\bigcup C' \subseteq X_0$.
- (4) If C' is a subchain of C_i such that both end links of C' intersect K , but no link of C' intersects H , then $\bigcup C' \subseteq X_2$.

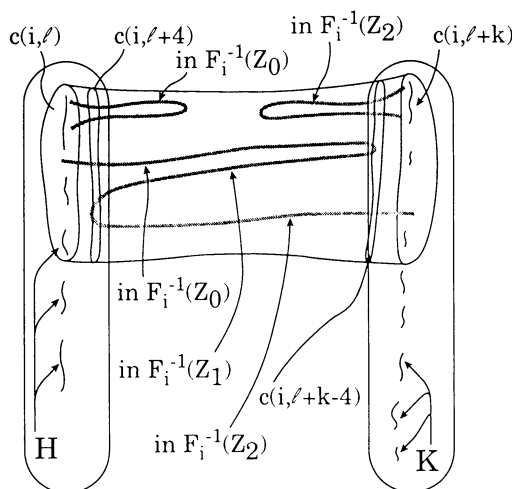


FIGURE 6

(5) Suppose C' is a subchain of C_i such that one end link of C' intersects H , the other end link intersects K , and no other link of C' intersects either H or K . Without loss of generality, let us say that $C' = \{c(i, l), \dots, c(i, l+k)\}$, $c(i, l) \cap H \neq \emptyset$, and $c(i, l+k) \cap K \neq \emptyset$. There is j , $0 \leq j \leq n_i - 1$, such that $Z = W_{i, n_i} \circ \dots \circ W_{i, n_i-j}(A_0) \cap F_i(\bigcup C')$ has exactly one component (in \mathbb{R}^2) that intersects both $F_i(c(i, l))$ and $F_i(c(i, l+k))$, and if $C'' = \{c(i, l+4), \dots, c(i, l+k-4)\}$, then $Z \cap F_i(\bigcup C'') = Z'$ has at least three components that intersect both $F_i(c(i, l+4))$ and $F_i(c(i, l+k-4))$. (See Figure 6. We are applying Lemma 4 here and using the one-component-three-component property it gives to get the “zigzag” needed between H and K to satisfy part III of the Krasiniewicz-Minc Theorem. One appropriate way to choose the “ n ” of Lemma 4 is to let n denote the greatest integer less than or equal to $k/2$, and then $j+1$ is the “ k ” of Lemma 4, and W_{i, n_i-j} corresponds to “ W_1 ” in Lemma 4, W_{i, n_i-j+1} to “ W_2 ”, \dots and W_{i, n_i} to “ W_k ”.) Construct sets Z_0 , Z_1 , and Z_2 as follows:

(a) If L is a component of Z intersecting $F_i(c(i, l))$, but not $F_i(c(i, l+k))$, then $L \subseteq Z_0$.

(b) If L is a component of Z intersecting $F_i(c(i, l+k))$, but not $F_i(c(i, l))$, then $L \subseteq Z_2$.

(c) If L is the component of Z intersecting both $F_i(c(i, l))$ and $F_i(c(i, l+k))$, then because of the “three component property” of Z' , L can be naturally divided into closed sets L_0, L_1, L_2 such that $L_0 \cap L_1 \subseteq F_i(\bigcup_{k=k-4}^k c(i, l+\hat{k}))$, $L_1 \cap L_2 \subseteq F_i(\bigcup_{k=0}^4 c(i, l+\hat{k}))$, $L_0 \cap L_2 = \emptyset$, $F_i(c(i, l)) \cap L \subseteq L_0$, $F_i(c(i, l+k)) \cap L \subseteq L_2$, and $L_0 \cup L_1 \cup L_2 = L$. Then $L_0 \subseteq Z_0$, $L_1 = Z_1$, and $L_2 \subseteq Z_2$.

Finally, $F_i^{-1}(Z_0) \subseteq X_0$, $F_i^{-1}(Z_1) \subseteq X_1$, and $F_i^{-1}(Z_2) \subseteq X_2$. Then X_0, X_1 , and X_2 are closed sets such that

- (6) $X_0 \cup X_1 \cup X_2 \supseteq X$,
- (7) $X_0 \cap X_1 \subseteq V$,
- (8) $X_1 \cap X_2 \subseteq U$,
- (9) $H \subseteq X_0$ and $K \subseteq X_2$, and

$$(10) X_0 \cap X_2 = \emptyset.$$

Thus by the Krasinkiewicz-Minc Theorem, X is hereditarily indecomposable. Since nondegenerate, hereditarily indecomposable, circularly chainable continua in \mathbb{R}^2 (or $S^1 \times \mathbb{R}^1$) are pseudocircles, X is a pseudocircle. \square

We define a *washer* to be a set homeomorphic to $S^1 \times [0, 1] \times [0, 1]$. Thus, a washer is just a set homeomorphic to a solid torus.

Theorem 6. *There are a C^1 -map $F_0: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\varepsilon > 0$, and disjoint copies R_1 and R_2 of $S^1 \times I \times I$ such that if $F \in N_\varepsilon^1(F_0)$, then*

- (1) $(F(R_1))^\circ \supseteq R_1 \cup R_2$ and $(F(R_2))^\circ \supseteq R_1 \cup R_2$,
- (2) F is a local diffeomorphism on $R_1 \cup R_2$ and is differentiable on \mathbb{R}^3 ,
- (3) $\mathbb{B} = \{(x, y, z) | F^n(x, y, z) \in R_1 \cup R_2 \text{ for each } n \geq 0\}$ is a continuous Cantor set of continua with $F(\mathbb{B}) = \mathbb{B}$, and
- (4) there is a dense G_δ -subset G of \mathbb{B} such that G is a union of components of \mathbb{B} and each component of G is a pseudocircle.

Proof. Let I_1 and I_2 be disjoint intervals in S^1 with $I_1 = [-\frac{1}{128}, \frac{41}{128}]$ and $I_2 = [\frac{63}{128}, \frac{100}{128}]$. Note that $\frac{1}{4} < |I_1| < \frac{1}{3}$ and $\frac{1}{4} < |I_2| < \frac{1}{3}$. Let $R_1 = S^1 \times [-2, 2] \times I_1$ and $R_2 = S^1 \times [-2, 2] \times I_2$.

For $z \in S^1$, let $g(z) = 3z \bmod 1$. Then $g|_{I_1}$ and $g|_{I_2}$ are one-to-one maps with $g(I_1^0)$ and $g(I_2^0)$ each containing both I_1 and I_2 . A standard argument then gives that if $\alpha_0, \alpha_1, \dots$ is a sequence of 1's and 2's, then there exists exactly one z in S^1 such that $g^i(z) \in I_{\alpha_i}$ for each $i \in \mathbb{N}$. Further, $C = \{z \in S^1 | g^n(z) \in I_1 \cup I_2 \text{ for each } n \in \mathbb{N}\}$ is a Cantor set in S^1 .

Define $F_0: S^1 \times \mathbb{R}^1 \times (I_1 \cup I_2) \rightarrow S^1 \times \mathbb{R}^1 \times S^1$ by $F_0(x, y, z) = (T(x, y), g(z))$ if $z \in I_1$ and $F_0(x, y, z) = (W^{-1}(x, y), g(z))$ if $z \in I_2$. Note that $F_1 = F_0|_{S^1 \times \mathbb{R}^1 \times I_1}$ is a local diffeomorphism, while $F_2 = F_0|_{S^1 \times \mathbb{R}^1 \times I_2}$ is a diffeomorphism. Choose $\varepsilon > 0$, $\varepsilon < 2^{-9}/\pi$, so that if $F \in N_\varepsilon^1(F_0)$, then (1) $F^{-1}|_{F(S^1 \times \mathbb{R}^1 \times I_2)} \in N_{2^{-7}/\pi}^1(F_2^{-1})$ and (2) if $\theta \subseteq F(S^1 \times [-2, 2] \times I_1)$ with $\text{diam } \theta < \frac{1}{2}$, then $F^{-1}|\theta \in N_{2^{-7}/\pi}^1(F_1^{-1}|\theta)$. (If θ is connected, then both $F_1^{-1}|\theta$ and $F^{-1}|\theta$ have exactly two components, each mapped homeomorphically onto θ by F_1 and F .)

Suppose $F \in N_\varepsilon^1(F_0)$. Checking that $(F(R_1))^\circ \supseteq R_1 \cup R_2$ and $(F(R_2))^\circ \supseteq R_1 \cup R_2$ is straightforward. Also, since F is in $N_\varepsilon^1(F_0)$, F is a local diffeomorphism on $R_1 \cup R_2$.

Remember that F_0 and F have been defined only on $S^1 \times \mathbb{R}^1 \times (I_1 \cup I_2)$ so far. We can visualize $F_0^{-1}(R_1)$ and $F_0^{-1}(R_2)$ as follows: Each (x, y, z) in R_1 has exactly three preimages in $S^1 \times \mathbb{R}^1 \times (I_1 \cup I_2)$. It has two preimages in R_1 and one in R_2 . The two preimages in R_1 occur because T is two-to-one and $(\frac{1}{2}x, \frac{1}{2}y)$ actually gives two points. The third coordinate $z/3$ is unique, so $(\frac{1}{2}x, \frac{1}{2}y, \frac{1}{3}z) = (T^{-1}(x, y), g^{-1}(z))$ corresponds to two points. There is only one preimage $(W(x, y), g^{-1}(z))$, since W is a diffeomorphism and $g|_{I_2}$ is one-to-one.

The set $F_0^{-1}(R_1)$ consists of two components, one component being in R_1 and the other in R_2 . Under F_0 slices in the z -direction get moved to slices in the z -direction.

The component of $F_0^{-1}(R_1)$ in R_1 is the set $S^1 \times [-1, 1] \times [-\frac{1}{3 \cdot 128}, \frac{41}{3 \cdot 128}]$,

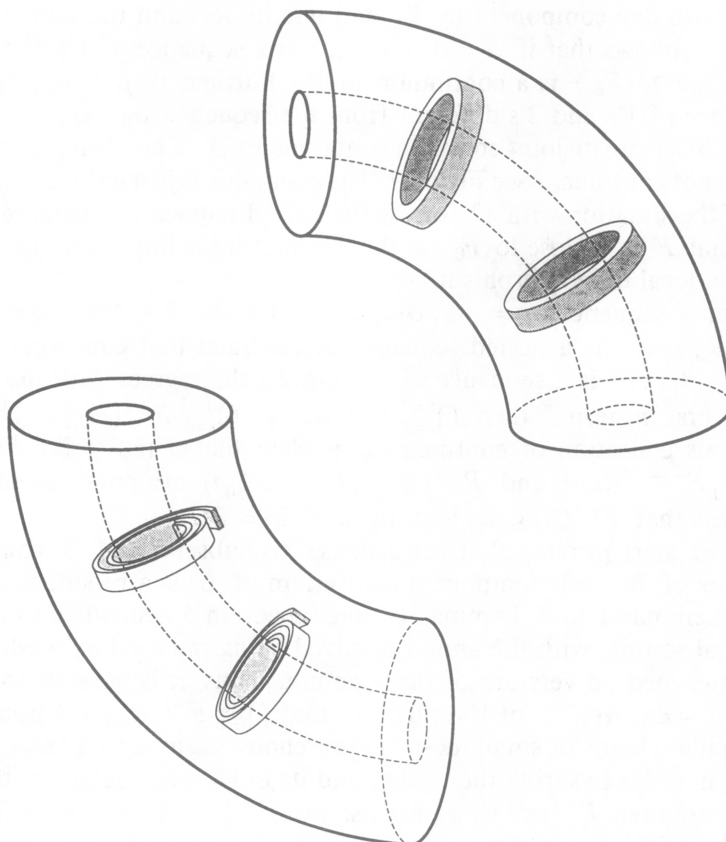


FIGURE 7

a washer as was R_1 . The component of $F_0^{-1}(R_1)$ in R_2 is $\{(W(x, y), \frac{1}{3}z + \frac{2}{3}) \mid (x, y, z) \in R_1\}$ and this set is just an annulus (a wiggly one; see Figure 7) crossed with an interval. Again we get a washer, although this one has wiggles. The component of $F_0^{-1}(R_1)$ in R_2 maps one-to-one onto R_1 and the component of $F_0^{-1}(R_1)$ in R_1 maps two-to-one onto R_1 . Note that with respect to the lift space $\mathbb{R}^2 \times S^1$, and appropriate three-dimensional strips of the form $J \times \mathbb{R} \times I_2$, the washer $F_0^{-1}(R_1) \cap R_2$ has the one-component, three-component property of Lemma 4. Each slice of $F_0^{-1}(R_1) \cap R_1$ is a set $S^1 \times [-1, 1] \times \{z\}$ and each slice of $F_0^{-1}(R_1) \cap R_2$ is a set $W(A_0) \times \{z\}$. As for $F_0^{-1}(R_2)$, the same discussion applies (with intervals in the z -direction adjusted), i.e., $F_0^{-1}(R_2)$ has two components, one in R_1 , which is just $S^1 \times [-1, 1] \times J$ for some interval J , and the other in R_2 of the form $W(A_0) \times J$ for some interval J . For $F \in N_e^1(F_0)$, the situation is the same, except that now some “warping” may occur. It is still the case that for $i = 1, 2$, $F^{-1}(R_i) \cap R_1$ is a washer close to $F_0^{-1}(R_i) \cap R_1$ and $F^{-1}(R_i) \cap R_2$ is a wiggly washer close to $F_0^{-1}(R_i) \cap R_2$, but, in particular, it is no longer the case that slices in the z -direction go to other slices in the z -direction. The slices may get warped a little, but topologically the situation is the same as before.

Consider $\mathbb{B} = \{(x, y, z) \in S^1 \times [-2, 2] \times (I_1 \cup I_2) \mid F^n(x, y, z) \in R_1 \cup R_2 \text{ for each } n \in \tilde{\mathbb{N}}\}$. Since $F^{-1}(R_1)$ has exactly two components, each of which maps

onto R_1 , with one component in R_1 and one in R_2 , and the same is true for $F^{-1}(R_2)$, it follows that if $\alpha = \alpha_0, \alpha_1, \dots$ is a sequence of 1's and 2's, then $B(\alpha) = \bigcap_{i=0}^{\infty} F^{-i}(R_{\alpha_i})$ is a continuum in \mathbb{B} . Further, if $\beta = \beta_0, \beta_1, \beta_2, \dots$ is a sequence of 1's and 2's different from the sequence $\alpha_0, \alpha_1, \alpha_2, \dots$, then $B(\alpha)$ and $B(\beta)$ are disjoint continua contained in \mathbb{B} . Therefore, \mathbb{B} consists of a Cantor set of continua. (See Figure 7. Note that this is just a three-dimensional version of the situation with C for F_0 that was discussed two paragraphs back. The fact that F is ε -close to F_0 (in the C^1 metric) is important, as is the fact that F is a local diffeomorphism.)

Now for a sequence $\alpha = \alpha_0, \alpha_1, \dots$ of 1's and 2's, the sequence $R_{\alpha_0}, \bigcap_{i=0}^{\infty} F^{-i}(R_{\alpha_i}), \dots$ is a nested sequence of continua that converges to $B(\alpha)$. If $\beta = \beta_0, \beta_1, \dots$ is a sequence of 1's and 2's that agrees with the sequence α for the first n terms, then $\bigcap_{i=0}^{n-1} F^{-i}(R_{\alpha_i}) = \bigcap_{i=0}^{n-1} F^{-i}(R_{\beta_i})$. Thus, \mathbb{B} is a continuous collection of continua. It is clear that $F(\mathbb{B}) \subseteq \mathbb{B}$. Since both $R_1 \cap (\bigcap_{i=0}^{\infty} F^{-i-1}(R_{\alpha_i}))$ and $R_2 \cap (\bigcap_{i=0}^{\infty} F^{-i-1}(R_{\alpha_i}))$ are continua in \mathbb{B} , it is also the case that $F^{-1}(\mathbb{B}) \subseteq \mathbb{B}$, and, thus, $F(\mathbb{B}) = \mathbb{B}$.

Before we start proving that for a dense G_δ -subset G of \mathbb{B} consisting of components of \mathbb{B} , each component continuum of G is a pseudocircle, please note that Lemmas 1 to 4, Lemma 4', and Theorem 5 generalize to our three-dimensional setting, with the annuli involved being replaced by washers which are just thickened up versions of those annuli. Thus, it is possible to choose a sequence $\alpha = \alpha_0, \alpha_1, \dots$ of 1's and 2's so that $\bigcap_{i=0}^{\infty} F^{-i}(R_{\alpha_i})$ is a pseudocircle. To get circular chains of small mesh we just choose as many 1's as we need at a given step in order to shrink the washer and its links, and double the number of links (because then $F^{-1}(x, y, z)$ is close to $(T^{-1}(x, y), g^{-1}(z))$). To get the desired level of the one-component—three-component property, we just choose as many 2's as we need. The 1's are used to make the resulting continuum circlelike and the 2's are used to make the small mesh circular chains "zigzag sufficiently," so that the continuum is hereditarily indecomposable in addition to being circlelike.

Suppose then that $E = e_0, e_1, \dots, e_n$ is an $(n+1)$ -element sequence of 1's and 2's. Let $l(E) = n$, and let $\hat{E} = \{(x, y, z) \in \mathbb{B} | F^i(x, y, z) \in R_1 \text{ if } e_i = 1, \text{ and } F^i(x, y, z) \in R_2 \text{ if } e_i = 2 \text{ for } 0 \leq i \leq l(E)\}$. Then $\mathcal{E} = \{\hat{E} | E \text{ is an } (n+1)\text{-term sequence of 1's and 2's for some } n \in \tilde{\mathbb{N}}\}$ is countable with \hat{E} both open and closed in \mathbb{B} .

For each pair $(n, \hat{E}) \in \mathbb{N} \times \mathcal{E}$, with $E = e_0, e_1, \dots, e_n$ there exist $m(n, \hat{E})$ and $k(n, \hat{E}) \in \mathbb{N}$ such that if, for each i , $1 \leq i \leq m(n, \hat{E})$, $1 = e_{\hat{n}+i}$, and for each i , $1 \leq i \leq k(n, \hat{E})$, $2 = e_{\hat{n}+m(n, \hat{E})+i}$, then $\hat{E} = \bigcap_{i=0}^{\hat{n}} F^{-i}(R_{e_i})$, and if $\hat{E}_T = \bigcap_{i=0}^{\hat{n}+m(n, \hat{E})} F^{-i}(R_{e_i})$ and $\hat{E}_{TW} = \bigcap_{i=0}^{\hat{n}+m(n, \hat{E})+k(n, \hat{E})} F^{-i}(R_{e_i})$, then

$$\mathcal{D} = \left\{ \bar{D} | D \text{ is a component of } \left(\bigcap_{i=0}^{m(n, \hat{E})+\hat{n}-1} F^{-i}(R_{e_i}) \right) \right. \\ \left. \cap F^{-\hat{n}-m(n, \hat{E})}((-1/4 \cdot 3/4) \times [-2, 2] \times I_1) \right\}$$

is a circular chain cover of \widehat{E}_T of mesh less than 2^{-n} , and (by an argument similar to that in the proof of Theorem 5) \widehat{E}_{TW} “zigzags sufficiently in \mathcal{D} .” (By “zigzags sufficiently” in \mathcal{D} , we mean the following: Let

$$\widetilde{\mathcal{D}} = \{\widetilde{\mathcal{D}} \mid \widetilde{\mathcal{D}} \text{ is a component (for some } \overline{D} \in \mathcal{D} \text{) of } \overline{D} \cap \widehat{E}_{TW}\}.$$

Thus, \widetilde{D} is a circular chain cover of \widehat{E}_{TW} . Suppose H and K are disjoint, closed sets of \widehat{E}_{TW} with the property that if d and d' are links of $\widetilde{\mathcal{D}}$ that intersect H and K , respectively, then d and d' are separated by at least 16 links. Further, suppose U and V are disjoint, open sets containing H and K , respectively, such that if d is a link of $\widetilde{\mathcal{D}}$ intersecting H or K , then the four links preceding d and the four links following d are all contained in U or V . As in Theorem 5, we may then construct the sets X_0 , X_1 , and X_2 satisfying the conditions of the Krasinkiewicz-Minc theorem from the links of $\widetilde{\mathcal{D}}$.)

Let

$$\begin{aligned} \widetilde{E}(n) = \{ & (x, y, z) \in \widehat{E} \mid F^i(x, y, z) \in R_1 \text{ for each } i \in \\ & \{l(E) + 1, \dots, l(E) + m(n, \widehat{E})\}, \text{ and } F^i(x, y, z) \in R_2 \\ & \text{for each } i \in \{1 + l(E) + m(n, \widehat{E}), \dots, l(E) + m(n, \widehat{E}) + k(n, \widehat{E})\} \}. \end{aligned}$$

Then for each n , $\widetilde{E}(n)$ is open in \widehat{E} , as well as closed, and $Z_n = \bigcup_{\widehat{E} \in \mathcal{E}} \widetilde{E}(n)$ is open and dense in \mathbb{B} .

Further, (x, y, z) is in Z_n implies that the component of \mathbb{B} that it is in has a circular chain cover (\mathcal{D}) of mesh less than 2^{-n} , and the circular chain refinement ($\widetilde{\mathcal{D}}$) for the circular chain cover (\mathcal{D}) “zigzags sufficiently” in the cover (\mathcal{D}). Then $\bigcap_{n=1}^{\infty} Z_n = G$ is a dense G_δ -set in \mathbb{B} , and $(x, y, z) \in G$ implies the component of \mathbb{B} containing (x, y, z) is a pseudocircle. Thus, we have our dense G_δ -subset of pseudocircles.

The only remaining problem is to do all this in \mathbb{R}^3 , rather than in $S^1 \times \mathbb{R}^1 \times S^1$. This is not difficult however. First embed $S^1 \times [-2, 2] \times I_1$ and $S^1 \times [-2, 2] \times I_2$ smoothly and disjointly in \mathbb{R}^3 . (One could perhaps think of these sets in \mathbb{R}^3 as being two pieces of elbow macaroni arcing about the origin.) Then it is possible to extend F_0 defined on these copies of $S^1 \times [-2, 2] \times I_1$, and $S^1 \times [-2, 2] \times I_2$ in a smooth way (using perhaps a version of the Tietze Extension Theorem), so that the resulting map on \mathbb{R}^3 has the properties desired. \square

IV. FUTURE GOALS

We hope to construct improved versions of this example. We would like to construct a perturbable diffeomorphism F with an extremely irregular invariant set many slices of which are generalized pseudocircles. This example will not be so easy to picture as the example constructed in this paper due to the increased number of dimensions, however.

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